

The Laplace Transformation

Let $f(t)$ be given and defined for $t \geq 0$,
Multiply $f(t)$ by e^{-st} and integrate with
respect to t from 0 to ∞ , is a function of S , Say
 $f(s)$;

$$f(s) = \int_0^{\infty} e^{-st} f(t) dt \quad \text{Laplace....} \quad (1)$$

Transform of $f(t) = \mathcal{L}(f)$

The operation is called the Laplace transformation

$$f(t) = \mathcal{L}^{-1}(F) = \text{inverse transform or inverse}$$

Theorem: Let $f(t)$ be a piecewise continuous on every finite interval in the range $t \geq 0$, and satisfies:

$$|f(t)| \leq Me^{\gamma t} \text{ for all } t \geq 0$$

and for some constants γ and M .

Then the Laplace transform exists for $s > \gamma$.

Proof :

$$|\mathcal{L}(f)| = \left| \int_0^{\infty} e^{-st} f(t) dt \right|$$

$$\leq \int_0^{\infty} e^{-st} |f(t)| dt$$

$$\leq \int_0^{\infty} e^{-st} M e^{\gamma t} dt$$

$$= M \int_0^{\infty} e^{-(s-\gamma)t} dt = \frac{M}{s-\gamma}$$

Examples :

$$|\mathcal{L}(1)| = \int_0^{\infty} 1 e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s} \quad \text{If } s > 0$$

$$\mathcal{L}(e^{at}) = \int_0^{\infty} e^{at} e^{-st} dt = \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^{\infty} = \frac{1}{s-a} \quad \text{If } s-a > 0$$

$$\mathcal{L}(e^{-at}) = \int_0^{\infty} e^{-at} e^{-st} dt = \frac{e^{-(s+a)t}}{-(s+a)} \Big|_0^{\infty} = \frac{1}{s+a} \quad \text{If } s+a > 0$$

$$\begin{aligned} \mathcal{L}(\cosh at) &= \frac{1}{2} \mathcal{L}(e^{at}) + \frac{1}{2} \mathcal{L}(e^{-at}) \\ &= \frac{1}{2} \left(\frac{1}{s-a} + \frac{1}{s+a} \right) = \frac{s}{s^2 - a^2} \quad \text{If } s > a \end{aligned}$$

$$\mathcal{L}(\sin wt) = \frac{w}{s^2 + w^2}$$

$$\mathcal{L}(\cos wt) = \frac{s}{s^2 + w^2}$$

$$\mathcal{L}(t^2) = \frac{2!}{s^3}$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \quad (n = 1, 2, 3, \dots)$$

$$\mathcal{L}(t) = \frac{1}{s^2}$$

$$\mathcal{L}(\sinh at) = \frac{a}{s^2 - a^2}$$

$$\mathcal{L}\left(\frac{1}{w^2}(1 - \cos wt)\right) = \frac{1}{s(s^2 + w^2)}$$

Set $a = i w$

$$\begin{aligned}\mathcal{L}(e^{at}) = \mathcal{L}(e^{iwt}) &= \frac{1}{s - iw} = \frac{s + iw}{s^2 + w^2} \\ &= \frac{s}{s^2 + w^2} + i \frac{w}{s^2 + w^2}\end{aligned}$$

But $e^{iwt} = \cos wt + i \sin wt$

$$\begin{aligned}\mathcal{L}e^{iwt} &= \mathcal{L} \cos wt + i \mathcal{L} \sin wt \\ &= \frac{s}{s^2 + w^2} + i \frac{w}{s^2 + w^2}\end{aligned}$$

$$\mathcal{L}(f^{(n)}) \\ = s^n \mathcal{L}(f) - s^{n-1} f_o - s^{n-2} f'_o - s^{n-3} f''_o \dots \dots \dots f_o^{(n-1)}$$

$$(f''') = s^3 \mathcal{L}(f) - s^2 f_o - s f'_o - f''_o$$

$$\mathcal{L}(f') = s \mathcal{L}(f) - f_o$$

Example: $\mathcal{L}(y) = \frac{s+1}{s^2+s-6}$, *find* $y(t)$

Solution:

$$\frac{s+1}{s^2+s-6} = \frac{s+1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

$$= \frac{A(s+3) + B(s-2)}{(s-2)(s+3)}$$

$s+1 = A(s+3) + B(s-2)$, *setting* $s=2$, and then $s=-3$

find $A = \frac{3}{5}$ and $B = \frac{2}{5}$

$$L\{y(t)\} = \frac{1}{5} \left(\frac{3}{s-2} + \frac{2}{s+3} \right)$$

$$\therefore y(t) = \frac{1}{5} (3e^{2t} + 2e^{-3t})$$

Example: $f(t) = \sin^2 t$, find $\mathcal{L}(f)$

Solution:

$$f'(t) = 2 \sin t \cos t = \sin 2t, \quad f(0) = 0$$

$$\mathcal{L}(\sin 2t) = \frac{2}{s^2 + 4},$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\frac{2}{s^2 + 4} = s\mathcal{L}(f) - 0$$

$$\therefore \mathcal{L}(f) = \frac{2}{s(s^2 + 4)}$$

Example: find the particular solution of
 $y'' - 3y' + 2y = 12e^{-2t} \quad y(0) = 2 \text{ \& } y'(0) = 6$

Solution:

$$\mathcal{L}y'' - 3\mathcal{L}y' + 2\mathcal{L}y = 12\mathcal{L}e^{-2t}$$

$$s^2\mathcal{L}y - sf_0 - f'_0 - 3(s\mathcal{L}y - f_0) + 2\mathcal{L}y = 12\mathcal{L}e^{-2t}$$

$$s^2\mathcal{L}y - s(2) - 6 - 3(s\mathcal{L}y - 2) + 2\mathcal{L}y = 12\frac{1}{s+2}$$

$$(s^2 - 3s + 2)\mathcal{L}y = 2s + \frac{12}{s+2} = \frac{2s^2 + 4s + 12}{s+2}$$

$$(s^2 - 3s + 2)\mathcal{L}y = \frac{2s^2 + 4s + 12}{s + 2}$$

$$\therefore \mathcal{L}y = \frac{2s^2 + 4s + 12}{(s^2 - 3s + 2)(s + 2)} = \frac{2s^2 + 4s + 12}{(s - 2)(s - 1)(s + 2)}$$

$$= \frac{A}{(s - 2)} + \frac{B}{(s - 1)} + \frac{C}{(s + 2)}$$

$$= \frac{7}{(s - 2)} - \frac{6}{(s - 1)} + \frac{1}{(s + 2)}$$

$$\therefore y(t) = 7e^{2t} - 6e^t + e^{-2t}$$

Example: Solve;

$$y'' + 4y' + 3y = 0 \quad y(0) = 3 \text{ \& } y'(0) = 1$$

Solution:

$$\mathcal{L}y'' + 4\mathcal{L}y' + 3\mathcal{L}y = 0$$

$$s^2\mathcal{L}y - sf_0 - f'_0 + 4(s\mathcal{L}y - f_0) + 3\mathcal{L}y = 0$$

$$s^2\mathcal{L}y - 3s - 1 + 4(s\mathcal{L}y - 3) + 3\mathcal{L}y = 0$$

$$(s^2 + 4s + 3)\mathcal{L}y = 3s + 1 + 12$$

$$\therefore \mathcal{L}y = \frac{3s + 13}{(s^2 + 4s + 3)} = \frac{3s + 13}{(s + 3)(s + 1)}$$

$$= \frac{A}{s + 3} + \frac{B}{s + 1}$$

$$= \frac{-2}{s + 3} + \frac{5}{s + 1}$$

$$\therefore y(t) = -2e^{-3t} + 5e^{-t}$$

Note:

$$\text{If } \mathcal{L}\{f(t)\} = F(s)$$

$$\text{Then; } \mathcal{L}\{e^{at}f(t)\} = F(s - a)$$

Example: Solve;

$$y'' + 2y' + 5y = 0 \quad y(0) = 2 \text{ \& } y'(0) = -4$$

Solution:

$$\mathcal{L}y'' + 2\mathcal{L}y' + 5\mathcal{L}y = 0$$

$$s^2\mathcal{L}y - sf_0 - f'_0 + 2(s\mathcal{L}y - f_0) + 5\mathcal{L}y = 0$$

$$s^2Y - 2s + 4 + 2(sY - 2) + 5Y = 0$$

$$Y = \mathcal{L}(y)$$

$$(s^2 + 2s + 5)Y = 2s$$

$$\therefore Y(s) = \mathcal{L}(y) = \frac{2s}{(s^2 + 2s + 5)} = \frac{2s}{(s + 1)^2 + 2^2}$$

$$= 2 \frac{s + 1}{(s + 1)^2 + 2^2} - \frac{2}{(s + 1)^2 + 2^2}$$

$$\text{But; } \mathcal{L}^{-1}\left(\frac{s}{s^2 + 2^2}\right) = \cos 2t \quad \& \quad \mathcal{L}^{-1}\left(\frac{2}{s^2 + 2^2}\right) = \sin 2t$$

$$\therefore y(t) = \mathcal{L}^{-1}(y) = e^{-t}(2 \cos 2t - \sin 2t)$$

Inverse Laplace Transforms:

$$\therefore \mathcal{L}\{f(t)\} = F(s)$$

$$\therefore f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Example:

$$\mathcal{L}(e^{2t}) = \frac{1}{s - 2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s + 2}\right\} = e^{-2t}$$

$$F(s)$$

$$\mathcal{L}^{-1}F(s) = f(t)$$

$$\frac{1}{s}$$

$$1$$

$$\frac{1}{s^2}$$

$$t$$

$$\frac{1}{s^{n+1}}$$

$$\frac{t^n}{n!}$$

$$\frac{1}{s-a}$$

$$e^{at}$$

$$\underline{F(s)}$$

$$\underline{\mathcal{L}^{-1}F(s) = f(t)}$$

$$\frac{1}{s^2 + a^2}$$

$$\frac{\sin at}{a}$$

$$\frac{s}{s^2 + a^2}$$

$$\cos at$$

$$\frac{1}{s^2 - a^2}$$

$$\frac{\sinh at}{a}$$

$$\frac{s}{s^2 - a^2}$$

$$\cosh at$$

Examples:

$$(1) \quad \mathcal{L}^{-1} \left(\frac{5}{s+3} \right) = 5e^{-3t}$$

$$(2) \quad F(s) = \frac{s+1}{s^2+2} \quad \text{find} \quad f(t)$$

$$F(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$\therefore f(t) = \cos t + \sin t$$

$$(3) \quad F(s) = \frac{1}{(s + 25)^2} \quad \text{find } f(t)$$

$$\text{Note: } \mathcal{L}\{e^{at}f(t)\} = F(s - a)$$

$$\mathcal{L}^{-1}\{F(s - a)\} = e^{at}f(t)$$

$$F(s) = \frac{1}{s^2}, \quad a = -25$$

$$f(t) = t$$

$\therefore$ The result is te^{-25t}

$$(4) \quad F(s) = \frac{s + 2}{(s + 2)^2 + 1} \quad \text{find } f(t)$$

$$f(t) = e^{-2t} \cos t$$

$$(5) \quad F(s) = \frac{s}{(s - 1)^2 - 4} \quad \text{find } f(t)$$

$$\frac{s - 1 + 1}{(s - 1)^2 - 4} = \frac{s - 1}{(s - 1)^2 - 4} + \frac{1}{(s - 1)^2 - 4}$$

$$\therefore f(t) = e^t \cosh 2t + \frac{1}{2} e^t \sinh 2t$$